JOURNAL OF COMPUTATIONAL PHYSICS 10, 232–241 (1972)

Solution of Troesch's Two-Point Boundary Value Problem by a Combination of Techniques

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Received January 6, 1972

Troesch's problem, which arose in the investigation of the confinement of a plasma column by radiation pressure, is an inherently unstable two-point boundary value problem. This paper discusses how Troesch's problem may be solved by a combination of methods, multipoint, continuation, and perturbation, although none of these methods by itself is sufficiently potent.

1. INTRODUCTION

Troesch's problem arose in the investigation of the confinement of a plasma column by radiation pressure [10]. In the version we shall study, it is the problem of solving the nonlinear ordinary differential equation

$$\ddot{y} = n \sinh ny, \qquad n > 1.0 \tag{1}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 1.0. \tag{2}$$

The associated initial value problem has a pole approximately at

$$t = (1/n) \ln(8/\dot{y}(0)),$$

a situation that makes the solution of (1), (2) by shooting methods very difficult. Even with an approximation quite close to the true value of the missing initial condition, $\dot{y}(0)$, it may be impossible to integrate (1) to the end of the interval [0, 1]. For example, in the case $n = 5$ for which the correct initial condition is $\dot{y}(0) \approx 0.038$, the reasonable guess $\dot{y}(0) = 0.055$ gives a solution to (1) that has a pole at $t < 1.0$. Numerical integration with $\dot{y}(0) = 0.055$ will cause overflow well before $t = 1.0$ is reached. Troesch's problem, due to its inherent instability, thus poses a severe test for any shooting method, and this severity increases with increasing n.

In the present paper we show how a combination of shooting methods, which individually may not be sufficiently powerful, can be used to solve Troesch's problem. It is to be expected that the combination of techniques, which has not received much attention in the literature (although Osborne [3] and Roberts and Shipman [9] have reported on a mixed strategy of multipoint and continuation techniques), will also be useful in solving other inherently unstable or numerically unstable two-point boundary-value problems.

An interesting approach to the solution of Troesch's problem by Monte Carlo methods described by Tsuda, Ichida, and Kiyono [11] should be noted. However, these authors do not attempt a solution for $n \geqslant 5.0$, while, as we will show, successful solutions at least up to and including $n = 10.0$ can be obtained by our combination of methods.

2. Combination of Methods

The methods to be combined are the perturbation technique [7], the multipoint or parallel shooting method [1, 2, 9], and the continuation method [4–6]. In the perturbation technique, the right hand side of the set of ordinary differential equations (written in the standard form $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$) is partitioned into two parts, usually linear and nonlinear. A perturbation parameter ϵ_k is introduced as a multiplier of one of the parts, usually the nonlinear terms. Starting with $\epsilon_0 = 0$, a sequence of two-point boundary-value problems is formed and solved. For the successive problems $\epsilon_{k+1} = \epsilon_k + \Delta\epsilon_k$, and the set of missing initial conditions found for the k-th problem is used as the initial trial set of values for the $(k + 1)$-st problem. When $\epsilon_k = 1.0$, the original problem will have been solved.

The multipoint (parallel shooting) method treats the given two-point boundary value problem as if it were, in fact, a multipoint boundary value problem. This is done by first dividing the original interval $[t_0, t_f]$ into Q subintervals $[t_{p-1}, t_p]$, $p = 1, 2, \ldots, Q$ with $t_Q = t_f$ and solving the original system of N equations over the Q subintervals simultaneously (in parallel) as a system of two-point boundary value problems. If the dimensionality of the original system is N, the dimensionality of the system resulting after the introduction of the multipoints is NQ. The N boundary conditions of the original system are supplemented by requiring that the solution of the multipoint system at the endpoint of the p-th interval be the same as the solution at the initial point of the $(p + 1)$-st interval; that is, continuity of the solution of the original system is invoked. Since there are $Q - 1$ interior multipoints, an additional $(Q-1)N$ boundary conditions are introduced making a total of $N+(Q-1)N=NQ$, precisely the necessary number. In exchange for the increased dimensionality, the multipoint method provides greater numerical stability and smaller error growth, as pointed out in [1].

The continuation technique is a method in which a parameter is varied continuously in some way to effect the solution [4–5]. In this sense, perturbation is a variant of continuation. The continuation technique applied here is a shooting method where the interval over which the boundary value problem is solved is considered as a parameter. Instead of attempting to solve the problem outright over the original interval $[t_0, t_f]$, the given problem is solved over a sequence of intervals $[t_0, t_p]$, where $t_p < t_{p+1}$ and $t_p \leqslant t_f$. For each problem of the sequence the prescribed boundary conditions of the original problem at t_f are taken as the boundary conditions at t_p, and the missing initial conditions found for the boundary value problem over the interval $[t_0, t_p]$ are used as the trial initial conditions for the boundary-value problem over the interval $[t_0, t_{p+1}]$. The process is terminated when $t_p = t_f$, at which point the original problem will have been solved .

Each of these three modified shooting techniques (perturbation, multipoint shooting, and continuation) has proven to be useful in the solution of numerically sensitive boundary value problems; that is, problems in which small changes in the initial conditions can precipitate numerical integration difficulties such as machine overflow or excessive error growth. By itself, however, none of these three was sufficiently powerful to solve Troesch's problem. In the next section we show how a combination of the methods can be used to solve Troesch's problem. This approach should be equally useful for other sensitive two-point boundary value problems.

3. Solution of Troesch's Problem

To solve Troesch's problem for a range of values of n we first converted the single second order differential Eq. (1) into an equivalent system of two first order equations by the standard substitutions $y = y_1$, $\dot{y} = y_2$. The boundary value problem (1), (2) then takes the form

$$\dot{y}_1 = y_2, \tag{3}$$

$$\dot{y}_2 = n \sinh ny_1, \tag{4}$$

$$y_1(0) = 0, \qquad y_1(1) = 1. \tag{5}$$

This preliminary transformation is a matter of convenience since our computer programs for solving two point boundary value problems assume that the system of equations is written in the standard form $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$, where $\mathbf{y}$ and $\mathbf{f}$ are $N \times 1$ vectors.

We started our attack on Troesch's problem by solving (3)–(5) for $n = 5$ over the interval [0.0, 1.0] using the multipoint method without continuation or perturbation. The selection of multipoints which ultimately led to the solution was obtained by trying to solve the problem successively with 2, 3, 6, and finally 9 multipoints. Each problem (except for the first) used the profiles from the previous problem as the source of the trial initial conditions at the multipoints. The multipoints themselves were concentrated near the end point $t = 1.0$, the region where overflow occurred.

TABLE Ia

$\ddot{y} = 5 \sinh 5y$

		2 multipoints		
t_p	$y_1^{(0)}(t)$	$y_2^{(0)}(t)$	$y_1^{(0)}(t)$	$y_2^{(0)}(t)$
0.00	0.0	$1.000(10^{-8})$	0.0	$1.0000000(10^{-8})$
0.25			$3.2037084(10^{-9})$	$1.8883734(10^{-8})$
0.50			$1.2099559(10^{-9})$	$6.1318909(10^{-8})$
0.75			$4.2493344(10^{-8})$	$2.1270193(10^{-7})$
1.00	1.0		$1.4838680(10^{-7})$	$7.4200137(10^{-7})$

TABLE Ib

3 multipoints

0.00	0.0	$1.000(10^{-8})$	0.0	$1.0000000(10^{-8})$
0.25			$3.2037084(10^{-9})$	$1.8883734(10^{-8})$
0.50			$1.2099595(10^{-8})$	$6.1318909(10^{-8})$
0.50	$1.209(10^{-8})$	$6.131(10^{-8})$	$1.2090000(10^{-8})$	$6.1310000(10^{-8})$
0.75			$4.2472370(10^{-8})$	$2.1260826(10^{-7})$
1.00	1.0		$1.4831718(10^{-7})$	$7.4165650(10^{-7})$

TABLE Ic

6 multipoints

0.00	0.0	$1.000(10^{-8})$	0.0	$1.0000000(10^{-8})$
0.50			$1.2100407(10^{-8})$	$6.1322887(10^{-8})$
0.50	$1.209(10^{-8})$	$6.131(10^{-8})$	$1.2090000(10^{-8})$	$6.1310000(10^{-8})$
0.75			$4.2473774(10^{-8})$	$2.1261526(10^{-7})$
0.75	$4.247(10^{-8})$	$2.126(10^{-7})$	$4.2470000(10^{-8})$	$2.1260000(10^{-7})$
0.80				
0.80	$4.247(10^{-8})$	$2.126(10^{-7})$	$4.2470000(10^{-8})$	$2.1260000(10^{-7})$
0.90			$7.0047245(10^{-8})$	$3.5038786(10^{-7})$
0.90	$4.247(10^{-8})$	$2.126(10^{-7})$	$4.2470000(10^{-8})$	$2.1260000(10^{-7})$
0.95			$5.4545189(10^{-8})$	$2.7292065(10^{-7})$
1.00	1.0		$7.0047245(10^{-8})$	$3.5038786(10^{-7})$

To be specific, for $n = 5$ the choice of multipoints was developed as follows. For two multipoints at $t = 0.0$ and 1.0, i.e., the original and final points, and the trial initial conditions $y_1^{(0)}(0) = 0.0$ and $y_2^{(0)}(0) = 10^{-8}$ (see columns 2 and 3 of Table Ia) a solution of the boundary value problem was attempted by the method of variational equations [8, 9]. The first iteration of the method yielded the values listed in columns 4 and 5 in Table Ia. However, the next iteration resulted in machine overflow. At this stage a multipoint was added at $t = 0.5$, and trial initial values for the resulting three-point boundary value problem obtained from columns 4 and 5 of Table 1a. With these values a solution was once more attempted by the method of variational equations, the first iteration yielding the profiles listed in columns 4 and 5 of Table Ib, while the second iteration again resulted in overflow. This process was repeated for six and finally nine multipoints, when a successful solution was obtained. Tables Ia, Ib, Ic and II show the location of the multipoints and the trial initial conditions at the multipoints. Columns 4 and 5 of Table Ic supplied the trial initial values at the multipoints for the nine multipoint problem exhibited in Table II, which also gives the solution for the tenth iteration of the method of variational equations for the problem $n = 5$. For the solutions at the

TABLE II

$\ddot{y} = 5 \sinh 5y$

t_p [a]	$y_1^{(0)}(t)$ [b]	$y_2^{(0)}(t)$	$y_1^{(10)}(t)$ [c]	$y_2^{(10)}(t)$ [c]
0.00	0.0	$1.000(10^{-8})$	0.0	$3.7766775(10^{-2})$
0.50			$4.5742875(10^{-2})$	$2.3230373(10^{-1})$
0.50	$1.209(10^{-8})$	$6.131(10^{-8})$	$4.5819097(10^{-2})$	$2.3354383(10^{-1})$
0.75			$1.6322359(10^{-1})$	$8.4004495(10^{-1})$
0.75	$4.247(10^{-8})$	$2.126(10^{-7})$	$1.6656719(10^{-1})$	$8.9852417(10^{-1})$
0.80			$2.1803689(10^{-1})$	1.1762980
0.80	$4.247(10^{-8})$	$2.126(10^{-7})$	$2.1937680(10^{-1})$	1.2386338
0.90			$3.8832871(10^{-1})$	2.3063291
0.90	$4.247(10^{-8})$	$2.126(10^{-7})$	$4.0646319(10^{-1})$	2.8455168
0.93			$5.0173755(10^{-1})$	3.5643220
0.93	$5.454(10^{-8})$	$2.729(10^{-7})$	$5.0680480(10^{-1})$	3.9419232
0.95			$5.9287259(10-1)$	4.7211609
0.95	$5.454(10^{-8})$	$2.729(10^{-7})$	$5.9635871(10^{-1})$	5.0963883
0.98			$7.7912400(10^{-1})$	7.4436956
0.98	$5.454(10^{-8})$	$2.729(10^{-7})$	$7.7099069(10^{-1})$	8.3569161
1.00	1.000		$9.9447534(10^{-1})$	$1.2730249(10^{1})$

[a] t_p, $p = 1, 2, \ldots$ = multipoints.

[b] $y_i^{(k)}(t)$ = value of $y_i(t)$ for the k-th iteration.

[c] Two values of $y_i^{(10)}$ are given at each t_p; the first is the result of the integration from t_{p-1} to t_p, the second is the initial value for the interval $[t_p, t_{p+1}]$.

multipoints more places of agreement can be achieved by more iterations of the process. As shown in Tables III–V, the multipoint process can give 6 to 7 decimal places of agreement at the multipoints.

TABLE III

$\ddot{y} = 6 \sinh 6y$

t_p	$y_1^{(0)}(t)$	$y_2^{(0)}(t)$	$y_1^{(10)}(t)$	$y_2^{(10)}(t)$
0.00	0.0	3.770(10⁻²)	0.0	1.7966041(10⁻²)
0.50			3.0015963(10⁻²)	1.8123198(10⁻¹)
0.50	4.581(10⁻²)	2.335(10⁻¹)	2.9999733(10⁻²)	1.8109896(10⁻¹)
0.75			1.3653736(10⁻¹)	8.4250957(10⁻¹)
0.75	1.665(10⁻¹)	8.985(10⁻¹)	1.3652211(10⁻¹)	8.4243492(10⁻¹)
0.80			1.8647773(10⁻¹)	1.1782968
0.80	2.193(10⁻¹)	1.238	1.8647716(10⁻¹)	1.1782781
0.90			3.6336876(10⁻¹)	2.6384734
0.90	4.064(10⁻¹)	2.845	3.6336874(10⁻¹)	2.6385112
0.93			4.5669138(10⁻¹)	3.6816420
0.93	5.068(10⁻¹)	3.941	4.5669105(10⁻¹)	3.6816205
0.95			5.4124245(10⁻¹)	4.8749013
0.95	5.963(10⁻¹)	5.096	5.4124227(10⁻¹)	4.8748919
0.98			7.3775052(10⁻¹)	9.0372454
0.98	7.709(10⁻¹)	8.356	7.3775065(10⁻¹)	9.0373446
1.00	1.000		1.0000016(10⁻¹)	2.0080680(10¹)

TABLE IV

$\ddot{y} = (n/2)(e^{\epsilon n y} - e^{-ny})$

$n = 10, \epsilon = 0.6$

t_p	$y_1^{(0)}(t)$	$y_2^{(0)}(t)$	$y_1^{(5)}(t)$	$y_2^{(5)}(t)$
0.00	0.0	1.645(10⁻³)	0.0	1.8250442(10⁻³)
0.50			8.8807916(10⁻³)	7.9001321(10⁻²)
0.50	2.198(10⁻¹)	1.579(10⁻¹)	8.8807920(10⁻³)	7.9001329(10⁻²)
0.75			7.9898470(10⁻²)	6.8915826(10⁻¹)
0.75	1.159(10⁻²)	1.657(10⁻¹)	7.9898484(10⁻²)	6.8915840(10⁻¹)
0.80			1.2279130(10⁻¹)	1.0526746
0.80	2.192(10⁻²)	2.718(10⁻¹)	1.2279129(10⁻¹)	1.0526742
0.85			1.8865048(10⁻¹)	1.6291696
0.85	4.732(10⁻²)	9.291(10⁻¹)	1.8865045(10⁻¹)	1.6291677
0.90			2.9292472(10⁻¹)	2.6552716
0.90	1.072(10⁻²)	1.287(10¹)	2.9292454(10⁻¹)	2.6552615
0.95			4.7637234(10⁻¹)	5.1374984
0.95	1.072(10⁻²)	1.287(10¹)	4.7637110(10⁻¹)	5.1374159
1.00	1.000		1.0000603	2.6017645(10¹)

TABLE V

$\ddot{y} = (n/2)(e^{\epsilon ny} - e^{-ny})$
$n = 10, \epsilon = 1.0$

t	$y_1^{(0)}(t)$	$y_2^{(0)}(t)$	$y_1^{(10)}(t)$	$y_2^{(10)}(t)$
0.00	0.0	$5.210(10^{-4})$	0.0	$3.6106045(10^{-4})$
0.10			$4.2431824(10^{-5})$	$5.5714508(10^{-4})$
0.25			$2.1844855(10^{-4})$	$2.2141236(10^{-3})$
0.50			$2.6792142(10^{-3})$	$2.6795376(10^{-2})$
0.50	$3.496(10^{-3})$	$3.406(10^{-2})$	$2.6792142(10^{-3})$	$2.6795376(10^{-2})$
0.60			$7.2838355(10^{-3})$	$7.2855353(10^{-2})$
0.75			$3.2713168(10^{-2})$	$3.2859253(10^{-1})$
0.75	$3.986(10^{-2})$	$3.882(10^{-1})$	$3.2713168(10^{-2})$	$3.2859252(10^{-1})$
0.80			$5.4143217(10^{-2})$	$5.4807001(10^{-1})$
0.80	$6.494(10^{-2})$	$6.361(10^{-1})$	$5.4143216(10^{-2})$	$5.4806997(10^{-1})$
0.85			$9.0225491(10^{-2})$	$9.3317237(10^{-1})$
0.85	$1.063(10^{-1})$	1.062	$9.0225488(10^{-2})$	$9.3317219(10^{-1})$
0.90			$1.5338461(10^{-1})$	1.6886960
0.90	$1.775(10^{-1})$	1.883	$1.5338460(10^{-1})$	1.6886949
0.95			$2.7910684(10^{-1})$	3.7895767
0.95	$3.159(10^{-1})$	4.124	$2.7910665(10^{-1})$	3.7895623
1.00	1.000		1.0002667	$2.5273774(10^{2})$

We next solved (3)–(5) for $n = 6$ as a nine-multipoint problem by the method of variational equations, using as the source of the trial initial values at the multipoints the solution for $n = 5$ in Table II. Table III lists the trial initial values and the solution at the tenth iteration of the method of variational equations for $n = 6$.

We then attempted to solve (3)–(5) for $n = 10$ as a nine multipoint problem, employing as a source of the trial initial values at the multipoints the solution for the problem for $n = 6$, but overflow occurred in the first iteration of the method of variational equations.

Since a combination of multipoint and continuation methods had been successful in the problems attacked in [3] and [9], we next tried this approach. We were able to solve the problem for $n = 10$ over the intervals [0.0, 0.90] and [0.0, 0.95], but we could not solve the problem over the interval [0.0, 1.0].

At this point we introduced a form of the perturbation scheme which we now describe. Ordinarily in applying the perturbation technique one attempts to partition the right hand side of the system of differential equations into linear and nonlinear parts. However, Troesch's equation does not lend itself to this approach. Instead, we replaced sinh ny by its definition

$$\sinh ny = \tfrac{1}{2}(e^{ny} - e^{-ny})$$

and introduced the perturbation parameter ϵ_k, $0 \leqslant \epsilon_k \leqslant 1$, into the exponent of e^{ny}, so that the differential equations now appear as

$$\dot{y}_1 = y_2, \tag{6}$$

$$\dot{y}_2 = \frac{n}{2}(e^{\epsilon_k n y_1} - e^{-n y_1}). \tag{7}$$

This was done in the expectation, borne out by subsequent analysis, that for $\epsilon_k < 1.0$, the solution of (6), (7) would grow at a slower rate than the solution of (3), (4), with the result that the troublesome pole of the solution would be moved out beyond the interval of interest [0.0, 1.0].

Setting $n = 10$, and $\epsilon_0 = 0.6$, and taking the trial missing initial conditions and the multipoints from the profiles of the seventh iteration of the solution of $\ddot{y} = 10 \sinh 10y$ over the interval [0.0, 0.95], we obtained the results presented in Table IV. Then, in succession, taking the trial missing initial conditions and the multipoints from the final iteration of the previous problem we solved (6), (7) (with the boundary conditions (5)), for $n = 10$ and $\epsilon_1 = 0.7$, $\epsilon_2 = 0.8$, $\epsilon_3 = 0.9$, and $\epsilon_4 = 1.0$. Thus we obtained a successful solution to Troesch's problem for $n = 10$, as exhibited in Table V.

As a rough check, the implicit solution possessed by Troesch's problem

$$t = \int_0^y dv/\sqrt{2 \cosh nv + C} \tag{8}$$

was computed by Romberg integration. In Eq. (8), the constant C, which is bounded from below by -2, is evaluated from the final condition $y(1) = 1.0$.

TABLE VI

Romberg Integration of $t = \int_0^y dv/\sqrt{2 \cosh nv + C}$
$n = 10$, $C = -1.9999998716$

t	y
0.00	0.0
0.10	$4.2084926(10^{-5})$
0.20	$1.3086482(10^{-4})$
0.30	$3.5781050(10^{-4})$
0.40	$9.7665595(10^{-4})$
0.50	$2.5138142(10^{-3})$
0.60	$7.4442756(10^{-3})$
0.70	$1.9371129(10^{-2})$
0.80	$5.3518334(10^{-2})$
0.90	$1.6076199(10^{-1})$
1.00	$9.9949024(10^{-1})$

Owing to the nature of the problem, the numerical quadrature will present some difficulties too, since the integrand has a pole at $t = 0$ when $C = -2$. In particular, as n increases, the evaluation of C becomes more sensitive. For example for $n = 5$, $C = -1.9979068951$, while for $n = 10$, $C = -1.9999998716$. The results of the Romberg integration are listed in Table VI.

4. Discussion

The use of a combination of techniques provided us with a great deal of flexibility in our attack on Troesch's problem. While our general approach was the systematic deformation of an initial "approximate" solution (which could be, at the start of the process, very far indeed from the true solution), our choice of options at various stages of this process was by no means unique. In particular, different choices could have been made in (1) the selection of the integration formula, (2) the development of the solution profiles, (3) the selection of the multipoints, (4) incrementing n, and (5) the partitioning scheme. These choices will now be justified in a brief discussion, which should also serve to give further insight into the combination of techniques and its potential advantages in dealing with sensitive problems.

(1) Integration formula. All our computations were executed in double precision arithmetic using a four point Runge–Kutta integration method, in order to use existing programs. More efficient formulas, such as Hamming's modification of Milne's formula, can be expected to give as good, if not better, results.

(2) Development of the profiles. When the second iteration of the method of variational equations applied to the two-point boundary value problem for $n = 5$ resulted in machine overflow, the authors turned to the multipoint method. An alternative choice would have been to try different values for the missing initial condition with the hope that one could be found which would lead to a successful solution. However, there is, in general, no systematic method of varying the trial initial conditions until a satisfactory set is found. Experience with the multipoint method, on the other hand, has resulted in rules-of-thumb which in practice enable a systematic attack on the problem, as the next paragraph indicates.

(3) Multipoint selection. The analyst can choose a different number of multipoints, different locations for the multipoints, and different values of the functions at the multipoints. To simplify the data input the authors have used the identical value of the function at several values of the multipoints in Table Ic and Table II for example.

As pointed out in [9], the authors have stated that the determination of the number of multipoints, the location of the multipoints, and the values of the functions at the multipoints is a matter of experimentation. The location of the multipoints is more important than the number of multipoints. In particular, the multipoints should be concentrated near those values of the independent variable where numerical problems occur.

(4) Incrementing n. The authors solved Troesch's problem for $n = 5$ and 6, and then to accelerate the process, attempted to solve the problem for $n = 10$. Successful solution for this last problem then exploited the continuation and perturbation techniques. An alternative strategy would have been to continue to increment n by smaller amounts, say 1.0 or 0.5.

(5) Partitioning scheme. In applying the perturbation scheme to the problem for $n = 10$, the authors employed a partitioning scheme different from that described in [7], since the form of the differential equation did not lend itself to the partitioning of the right hand side into linear and nonlinear terms. The manner of introducing the perturbation parameter ϵ remains something of an art, although the authors attempt to systematize the process in a forthcoming paper. The choice of the $\Delta\epsilon$ increment is another option which can be exericsed.

With all the flexibility provided by the combination of techniques, we believe that other inherently unstable or numerically unstable two-point boundary value problems can be successfully attacked by this approach.

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